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ON GENERATING ELEMENTS OF SIMPLE RING EXTENSIONS

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Throughout the present note, A will be a simple (artinian) ring with the identity element 1 which is represented as $\sum_1^n De_{ij}$ with a system of matrix units $E = \{e_{ij}\}$ such that the centralizer $V_A(E)$ of E in A is the division ring D . In subsequent use, a subring of A will mean one containing the identity element 1 of A . Further, we use the following conventions: B will mean a simple subring of A , and \mathfrak{G} the group of all ring automorphisms in A which leave every element of B invariant. The centers of A and B will be denoted by C and Z respectively, and we use the notations $V = V_A(B)$, $H = V_A(V)$ and $C_0 = V \cap H (= \text{center of } V)$. If the fixring of \mathfrak{G} coincides with B and V is simple, A/B is said to be Galois (cf. [6]).

Concerning generating elements of a finite Galois extension A/B , we have obtained a number of interesting results ([2], [3] and [9]), and seen that the tools used in the respective cases $[B:Z] = \infty$ and $[B:Z] < \infty$ are strikingly distinct. In this note, we shall sharpen slightly the key propositions for these respective cases which were given in [2] and [3], and give as easy consequence of these sharpenings several results obtained previously in [2], [3], [4] and [9] with some refinement.

1.

In this section, the preliminary results will be stated without proof as lemmas.

Lemma 1 ([8; Lemma]). *If $[A:C] < \infty$ then $[B:Z] < \infty$. The converse is true, provided A is left (or right) finite over B .*

Lemma 2 ([1; Theorem VII. 11. 3] or [7]). *Let A be a division ring, and $[A:C] < \infty$. If M is a maximal subfield of A and $M = C[m]$ then there exists an element a such that $A = \sum_{i,j=0}^t m^i a m^j C$.*

Lemma 3 ([6] and [2; Lemma 1.4]). *Assume that A is finite Galois over B .*

(a) *If B' is a simple intermediate ring of A/B and $V_A(B')$ is simple then A is Galois over B' .*

(b) If V is a division ring then every intermediate ring of A/B is simple, and conversely.

Lemma 4 ([2; Corollary 2.1]). Assume that A is finite Galois over B and $[B:Z]=\infty$. If X is a B - B -submodule of A then $X=BxB$ with some x . In particular, every intermediate ring of A/B is singly generated over B .

Lemma 5 ([3; Lemma 7]). Let $n \geq 2$, and let x be a non-zero element of D . If a is an element of A not contained in C then there exists a unit r of A such that $a\tilde{r}=rar^{-1}=\sum x_{ij}e_{ij}$ where $x_{in}=x$ and $x_{in}=0$ for every $i \geq 1$.

Lemma 6 ([5; Lemma 1]). Let $n \geq 2$, and let T be a left artinian subring of A . If T contains $a=\sum x_{ij}e_{ij}$ ($x_{ij} \in D$) such as $x_{in}=1$ and $x_{in}=0$ for every $i \geq 1$ and $u(E, d)=de_{n1}+\sum_{i=1}^n e_{ii-1}$ with non-zero $d \in D$, then T is a simple ring containing E , d and x_{ij} 's.

2.

We shall prove first the key proposition for the case $[B:Z]=\infty$.

Proposition 1. Assume that A is finite Galois over B and $[B:Z]=\infty$.

(a) If A' is an intermediate ring of $A/B \cdot V$ then $A'=B[a']$ with some unit a' . In particular, $A=B[a]$ with some unit a .

(b) If V is a division ring, then for every intermediate ring A' of A/B there exists a unit a' of A' such that $A'=B[a']$.

Proof. (a) As $B \cdot V = B \otimes_Z V$ is simple and $V_A(B \cdot V) = C_0$, A' is a simple ring by Lemma 3. Accordingly, we have $A' = \sum_1^{n'} D' e'_{ij}$ where $E' = \{e'_{ij}\}$ is a system of matrix units such that $V_{A'}(E')$ is the division ring D' . Obviously, by Lemma 4, it suffices to prove the case $n' \geq 1$. Moreover, in virtue of Lemma 5, we may assume that B contains an element $b = \sum x'_{ij} e'_{ij}$ ($x'_{ij} \in D'$) such as $x'_{in}=1$ and $x'_{in}=0$ for every $i \geq 1$. We consider here the simple ring $B_1 = B[E'] = \sum_1^{n'} D_1 e'_{ij}$, where $D_1 = V_{B_1}(E')$ is a division subring of D' . If we set $V = \sum_1^r U g_{pq}$ with a system of matrix units $\Gamma = \{g_{pq}\}$ such that $U = V_\Gamma(\Gamma)$ is a division ring, then we may assume further that E' contains Γ . It follows then $V_A(B_1) = V_\Gamma(E') = V_\Gamma(E')$ is a division ring, and therefore $V_A(E')$ is finite Galois over D_1 by Lemma 3 (a). Since D_1 is infinite over its center (Lemma 1), we can find a non-zero element d' such that $D' = D_1[d']$ (Lemma 4). Setting $a' = 1 - u(E')$,

$d')$, a' is a unit of A' and Lemma 6 proves $A' = B[b, u(E', d')] = B[a']$.

(b) As V is a division ring, A' is simple by Lemma 3 (b) and the proof proceeds in the same way as in (a).

3.

The next is stated in [9]. However, for the sake of completeness, we shall give here the proof.

Theorem 1. *If A is a separable simple algebra of finite rank over a field ϕ then $A = \phi[u, u\tilde{r}]$ with some units u and r .*

Proof. Case I. $n=1$: As is well known, A contains a maximal subfield M which is separable over ϕ . Since $M = \phi[u]$ with some u , by Lemma 2 there exists a unit r such that $A = \sum_{i,j} u^i r u^j C = \sum u^i (u\tilde{r})^j C = \phi[u, u\tilde{r}]$.

Case II. $n > 1$: As D is a separable division algebra over ϕ , by Case I there exist non-zero elements $x, d \in D$ such that $D = \phi[x, x\tilde{d}]$. We set $t = \sum_1^n e_{in-t+1} (= t^{-1})$, $u^* = u(E, 1)$ and $v^* = u^* \tilde{t} = \sum_1^n e_{i-1t}$. Then, one will easily see that $e_{ij} = u^{*i-1} v^{*n-1} u^{*n-1} v^{*j-1}$ ($i, j = 1, 2, \dots, n$). In case $D = \phi$, $1 - u^*$ is a unit and $\phi[1 - u^*, (1 - u^*)\tilde{t}] = \phi[u^*, v^*] = \phi[E] = A$. Thus, in what follows, we may restrict our attention to the case $D \neq \phi$. Under this situation, if $x\tilde{d} \cdot x = 1$ then $\phi[x, x\tilde{d}] = \phi[x, x\tilde{1}]$ and $x \cdot x \neq 1$. Accordingly, we may assume further that $x\tilde{d} \cdot x \neq 1$. If $u = u^* + x e_{1n}$ and $v = u\tilde{d}t = v^* + x\tilde{d} \cdot e_{n1}$ then $u^{-1} = v^* + x^{-1} e_{n1} \in \phi[u]$ yields $(x^{-1} - x\tilde{d})e_{n1} = u^{-1} - v \in \phi[u, v]$, whence it follows $(1 - x\tilde{d} \cdot x)e_{nn} = (u^{-1} - v)u \in \phi[u, v]$. Noting that $x = u^n$ and $x\tilde{d} = v^n$ are contained in $\phi[u, v]$, we obtain $e_{nn} \in \phi[u, v]$, which forces $e_{ij} = v^{n-i} e_{nn} u^{n-j} \in \phi[u, v]$. Consequently, we have $\phi[u, v] = \phi[x, x\tilde{d}, E] = A$, completing the proof.

Corollary 1. *If A is finite Galois over B and $[B:Z] < \infty$ then $A = (B \cap C)[u, u\tilde{r}]$ with some units u and r .*

Proof. Since $[A:C] < \infty$ by Lemma 1 and C is finite Galois over $B \cap C$, A is a separable simple algebra of finite rank over $B \cap C$.

4.

The next is the key result for the case $[B:Z] < \infty$ (cf. [3; Proposition 1]).

Proposition 2. *Assume that A is finite Galois over B and $[B:Z]$*

$< \infty$. Let A^* be a simple intermediate ring of A/B such that the center C^* of A^* is contained in C_0 .

(a) If A^* is commutative then $A^* = Z[c_0]$ with some c_0 .

(b) If a is an arbitrary element of A^* not contained in C^* then there exists a unit a' of A^* such that $A^* = Z[a, a']$.

Proof. Let $\phi = B \cap C$ and $A^* = \sum_1^{n^*} D^* e_{ij}^*$ where $E^* = \{e_{ij}^*\}$ is a system of matrix units such that $V_{A^*}(E^*)$ is the division ring D^* . In the proof of [3; Proposition], we must complete only the case that $D^* \neq C^*$ and $n^* = 2$. In any rate, we have known that $A^* = Z[a, a']$ with some a' . Noting that ϕ is infinite, one will easily see that there exists an element $\alpha \in \phi$ such that $a' = a'' - \alpha$ is a unit of A^* . Then, we obtain $A^* = Z[a, a''] = Z[a, a']$.

5.

As the first consequence of Proposition 2, we obtain the following [4; Theorem 1] with an extremely short proof (cf. also [3; Theorem 1]).

Theorem 2. *Let A be a separable simple algebra of finite rank over a field ϕ . If a is an arbitrary element of A not contained in C then $A = \phi[a, a']$ with some unit a' .*

Proof. As is well known, the central simple algebra A over C has a finite Galois extension C^* of C as a splitting field, where we may assume further C^* is Galois over ϕ . Then, $A^* = A \otimes_C C^* = (C^*)_m$ is finite Galois over ϕ , and so we can apply Proposition 2 to A^*/ϕ and A to see that there exists a unit a' such that $A = \phi[a, a']$.

Next, combining Proposition 1 with Proposition 2 and Corollary 1, one will obtain at once the following sharpening of [3; Theorem 5] and [3; Corollary 2] itself.

Theorem 3. *Assume that A is finite Galois over B .*

(a) *If a is an arbitrary element of A not contained in C then there exists a unit a' such that $A = B[a, a']$. Accordingly, A/B is singly generated if and only if either $A = C$ or $B \subsetneq C$.*

(b) *$A = B[u, u\bar{r}]$ with some units u and r .*

Moreover, the validity of Proposition 1 enables us to see the following (cf. [3; Theorem 2]):

Theorem 4. *Assume that A is finite Galois over B . If V is commutative then for every intermediate ring A' of A/B there exists a unit a' such that $A' = B[a']$.*

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